Corrections for the paper Surgeries on periodic links and homology of periodic 3-manifolds

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As was first pointed out by Qi Chen in his email to the authors, Remark 2·4 in our last paper [1] is wrong. As one of the counterexamples, consider the orbitally separated link L^p with a diagram in Fig. 1c. It is easy to check that the corresponding underlying link L_* is algebraically split, nevertheless the linking number between the components l_{11} and l_{21} is not zero.

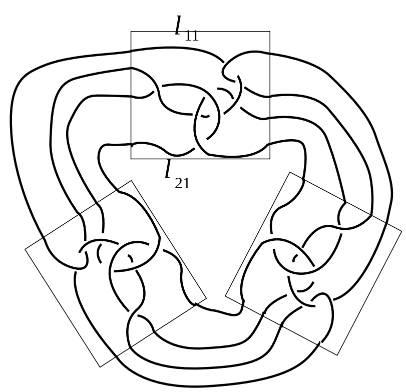

Fig. 1c

As Remark 2·4 does not hold, and so Corollary 2·5, the proof of Proposition 2·10 (which is the main building block of Theorem 2·1) needs to be changed. First of all, let us replace the defaulted Remark 2·4 and Corollary 2·5 by somewhat weaker Remark 2·4c and Corollary 2·5c.

Remark 2·4c. A strongly p-periodic link L^p is orbitally separated if and only if for any two components l_{ij} and l_{ks} of L^p that cover different components of L_* the linking number between l_{ij} and the orbit of l_{ks} is equal to zero (i.e., $\mathrm{lk}(l_{ij}, l_{k1}) + \mathrm{lk}(l_{ij}, l_{k2}) + \ldots + \mathrm{lk}(l_{ij}, l_{kp}) = 0$).

COROLLARY 2·5C. *It follows from Remark 2.4c that L^p is an orbitally separated link if and only if for every non-diagonal block B_{ij} in the matrix A_p (see section 2·1 of [1]) the sum of elements in any column (or any row) of B_{ij} is zero.* □

In order to prove Proposition 2·10, we will need three more matrix-theoretical results. They can be considered as generalizations of Lemma 2·6 and Lemma 2·7.

LEMMA 2·6·1. *Let A_p be the linking matrix for an orbitally separated strongly periodic link L^p (as constructed in section 2·1 of [1]), then*

$$detA_p = detA_{11}\, detA_{22} \ldots detA_{nn} \pmod p$$

Proof. We will use essentially the same ideas as in the proof of Lemma 2·6 of [1]. The determinant of A_p is a sum of $(np)!$ terms. Recall, that by Proposition 2·3, every block A_{ii}, B_{ij}, $i, j \in \{1, 3, \ldots, n\}$ is a circulant matrix. We will say that a term $\mathbf{x}$ of $\det A_p$ is *block-diagonal* if it can be represented in the following form:

$$\mathbf{x} = \prod_{i,j=1}^{n} (x^{ij}_{k(ij)})^p,$$

where $x^{ij}_{k(ij)}$ is the $(1, k(ij))$-th element of the block B_{ij}, if $i \neq j$, or A_{ii}, if $i = j$, for some $1 \leq k(ij) \leq p$.

1) First, we will show that $\det A_p$ is modulo p equal to the sum of its block-diagonal terms. Indeed, consider any term

$$\mathbf{y} = \prod_{i=1}^{np} y_{i\sigma(i)}.$$

Here we follow the convention that y_{ij} is the (i, j)-th element of A_p, and σ is a permutation on the set $\{1, 2, \ldots, np\}$. By Φ we will denote a map on the set of all terms of $\det A_p$, defined as follows:

$$\Phi\mathbf{y} = \prod_{i=1}^{np} y_{\phi(i)\phi\sigma(i)},$$

where

$$\phi(m) = \begin{cases} m+1, & \text{if } p \not| m \\ m-p+1, & \text{if } p|m. \end{cases}$$

For instance, if $p = 3$, $n = 2$, and $\mathbf{y}= y_{12}y_{21}y_{35}y_{44}y_{56}y_{63}$, then $\Phi\mathbf{y} = y_{23}y_{32}y_{16}y_{55}y_{64}y_{41}$.

Notice, that $\mathbf{y}$ and $\Phi\mathbf{y}$ represent the same term in $\det A_p$ if and only if $\mathbf{y}$ is block-diagonal. Otherwise, since p is prime, $\mathbf{y}$, $\Phi\mathbf{y}, \Phi^2\mathbf{y}, \ldots, \Phi^{p-1}\mathbf{y}$ all represent different terms in $\det A_p$. On the other hand, as numbers modulo p, they all are equal, which follows from the fact that A_p consists of n^2 circulant blocks of size $p \times p$. Moreover, all p of the above terms have the same sign in the sum $\det A_p$. Indeed, the sign of a term in a determinant is defined by the parity of the permutation mapping the row indices into the corresponding column indices. Thus, it is enough if we show that the permutations σ and

$$\begin{pmatrix} \phi(1) & \phi(2) & \cdots & \phi(np) \\ \phi\sigma(1) & \phi\sigma(2) & \cdots & \phi\sigma(np) \end{pmatrix}$$

have the same parity, which easily follows from the fact that the map ϕ is one-to-one. Therefore, if $\mathbf{y}$ is not a block-diagonal term of $\det A_p$, then the sum of the terms $\mathbf{y}, \Phi\mathbf{y}, \Phi^2\mathbf{y}, \ldots, \Phi^{p-1}\mathbf{y}$ is equal to zero modulo p.

2) We have shown above that $\det A_p$ is modulo p equal to the sum of its block-diagonal terms. Each block $B_{i,j}$ of A_p is a circulant matrix, and therefore, by Lemma 2·6, $\det B_{ij}$ is modulo p equal to the sum of its diagonal terms. These two facts together imply that

$$\det A_p = \sum_{\sigma \in S_n} (\text{sgn } \sigma) \det B_{1\sigma(1)} \ldots \det B_{n\sigma(n)} \pmod{p},$$

where $B_{kk} = A_{kk}$. By Corollary 2·5c, for every B_{ij} such that $i \neq j$, we have $\det B_{ij} = 0$. Therefore, $\det A_p = \det A_{11} \ldots \det A_{nn} \pmod{p}$. □

Before we formulate the next result, let us introduce some notations. For a matrix A by RA_i we will denote the i-th row of A, and by CA_i we will denote the i-th column of A. We will use the following notations for elementary transformations of A:

$RA_i \mapsto kRA_i$ means multiplication of the i-th row by $k \neq 0 \pmod p$,

$RA_i \mapsto RA_i + kRA_j$ means replacement of the row RA_i by the linear combination $RA_i + kRA_j$.

These notations can be generalized to sequences of elementary transformations. Thus, $RA_i \mapsto \sum k_j RA_j$ means that we replace RA_i by the linear combination $\sum k_j RA_j$, which can be achieved by a sequence of elementary transformations, given $k_i \neq 0 \pmod p$. Similar notations will be used for elementary transformations of the columns.

The following technical result was essentially presented as a part of the proof of Lemma 2·7 of [1], but here we will have it in a little more general form.

PROPOSITION 2·7·1. *1) Let p be a prime number, and let*

$$A = \begin{pmatrix} a_1 & a_2 & a_3 & \cdots & a_p \\ a_p & a_1 & a_2 & \cdots & a_{p-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_2 & a_3 & a_4 & \cdots & a_1 \end{pmatrix}$$

be a circulant matrix with $detA \neq 0 \pmod p$. *Then the following sequence of elementary transformations*

(a) $RA_p \mapsto \sum_{i=1}^{p} RA_i$,

(b) $RA_{p-1} \mapsto \sum_{i=1}^{p-1} (p-i) RA_i$,

performed modulo p, turns A into the following equivalent matrix:

$$A' = \begin{pmatrix} a_1 & a_2 & a_3 & \cdots & a_p \\ a_p & a_1 & a_2 & \cdots & a_{p-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ x & x & x & \cdots & x \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}$$

2) If A is symmetric, then $x = 0 \pmod{p}$.

Proof. Since $\det A \neq 0 \pmod{p}$, then by Lemma 2·6, we have $a_1 + a_2 + \ldots + a_p = 0 \pmod{p}$. Therefore, the first sequence of elementary transformations creates zeros in the last row. Now, let us consider the transformation (b). First, we need to show that $S = \sum_{i=1}^{p-1}(p-i)RA_i = (x, x, \ldots, x) \pmod{p}$, where

$$x = (p-1)a_p + (p-2)a_{p-1} + \ldots + a_2.$$

Indeed, the i-th coordinate of the vector S is

$$(p-1)a_i + (p-2)a_{i-1} + \ldots + 2a_{i+3} + a_2,$$

with all coefficients and subscripts treated modulo p. After the substitution $a_1 = -a_2 - a_3 - \ldots - a_{p-1} - a_p$, the coefficient for $a_k, k \neq 1$, in the above sum is

$$(p-1-(i-k))-(p-i)=k-1 \pmod p,$$

exactly as in the linear combination x. This proves the first half of the proposition.

Now, if A is symmetric, then $a_i = a_{p-i+2}$, all indices modulo p, $i \neq 1 \pmod p$. Then, in the linear combination x the coefficient for a_i is $(p-1-(p-i))+(p-1-(i-2)) = p = 0 \pmod p$. □

The next lemma is an important generalization of Lemma 2·7 from [1].

LEMMA 2·7·2. *Let p be an odd prime integer, and let A_p be the linking matrix for an orbitally separated strongly periodic link L^p, then $null_p A_p \neq 1$.*

Proof. Assume that the corresponding underlying link L^* has n components. By Proposition 2·3 and Corollary 2·5c, A_p can be represented as follows:

$$A_p = \begin{pmatrix} A_{11} & B_{12} & \cdots & B_{1n} \\ B_{21} & A_{22} & \cdots & B_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ B_{n1} & B_{n2} & \cdots & A_{nn} \end{pmatrix},$$

where every block among A_{ii} and B_{ij} is a $p \times p$ circulant matrix, the diagonal blocks $A_{11}, \ldots, A_{nn}$ are symmetric (because A_p is a linking matrix), and all non-diagonal blocks B_{ij}, $i, j = 1, \ldots, n$, $i \neq j$, are degenerate modulo p.

If $n = 1$, then this lemma follows from Lemma 2.7 of [1]. Assume $n \geq 2$. Then we have one of the following three cases.

1). If $\det A_{11} \det A_{22} \ldots \det A_{nn} \neq 0 \pmod{p}$, then $\det A_p \neq 0 \pmod{p}$, by Lemma 2·6·1. Therefore, $\mathrm{null}_p A_p = 0 \neq 1$.

2). If there are at least two degenerate modulo p diagonal blocks, then without loss of generality, we can assume that $\det A_{11} = \det A_{22} = 0 \pmod{p}$. By Lemma 2·6 of [1], this implies that the following sequences of elementary transformations

$$R(A_p)_p \mapsto \sum_{i=1}^{p} R(A_p)_i,$$

$$R(A_p)_{2p} \mapsto \sum_{i=p+1}^{2p} R(A_p)_i,$$

create zeros modulo p out of all elements of the p-th and $2p$-th rows of A_p. Thus, $\mathrm{null}_p A_p \geq 2$.

3). We are left with the most interesting case: without loss of generality, let us assume that $detA_{11} = 0 \pmod{p}$, and $\det A_{22} \ldots \det A_{nn} \neq 0 \pmod{p}$. Let us apply to A_p the sequence of elementary transformations from Lemma 2·7·1:

$$R(A_p)_p \mapsto \sum_{i=1}^{p} R(A_p)_i,$$

$$R(A_p)_{p-1} \mapsto \sum_{i=1}^{p-1} (p-i) R(A_p)_i.$$

As was shown in Lemma 2·7·1, as the result of these transformations, the last two rows of A_{11} will become zeros, and every non-diagonal block B_{1k},

$k = 2, \ldots, n$, will take the following form:

$$B_{1k} \to \begin{pmatrix} * & * & * & \cdots & * \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ * & * & * & \cdots & * \\ x_k & x_k & x_k & \cdots & x_k \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}$$

If all of the numbers $x_2, \ldots, x_k$ are zeros modulo p, then we automatically have the result. Let us assume, that one of them is not zero. Without loss of generality, we can assume that $x_2 \neq 0 \pmod p$. Add the rows $R(A_p)_{p+1}, \ldots, R(A_p)_{2p-1}$ to the row $R(A_p)_{2p}$. By Lemma 2·6 of [1], all the elements of the $2p$-th row, belonging to A_{22}, will turn into $\alpha = \det A_{22} \neq 0 \pmod p$, and all the other elements of that row will turn into zeros. Now we need to get rid of all the α's. To do that, let us first subtract the column $C(A_p)_{p+1}$ from each of the columns $C(A_p)_{p+2}, \ldots, C(A_p)_{2p}$. After that operation, the only non-zero elements of the rows $R(A_p)_{p-1}$ and $R(A_p)_{2p}$ will be x_p and α correspondingly, both belonging to the column $C(A_p)_{p+1}$. The last transformation

$$R(A_p)_{2p} \mapsto R(A_p)_{2p} - \alpha x_k^{-1} \cdot R(A_p)_{p-1}$$

turns $R(A_p)_{2p}$ into a zero row. Thus, $\mathrm{null}_p A_p \geq 2$. □

Now we are ready to fix the proof of of Proposition 2·10 from [1]:

PROPOSITION 2·10. *Let p be an odd prime integer. If a closed orientable 3-manifold M can be obtained from S^3 by a Dehn surgery on an orbitally separated framed link L^p then $H_1(M;\mathbf{Z}_p) \neq \mathbf{Z}_p$.*

Proof. Assume that M can be obtained by Dehn surgery on an orbitally separated framed link L^p. Let A_p be the linking matrix of L^p, as constructed in Section 2.1 of [1]. By Corollary 2·5c and Lemma 2·7·2, $null_p A_p \neq 1$. Hence, by Lemma 2·9 of [1], we have $H_1(M;\mathbf{Z}_p) \neq \mathbf{Z}_p$. □

Since Proposition 2·10 holds, the proof of Theorem 2·1 of [1] also stays intact. Unfortunately, this does not fix the proof of Theorem 2·2. We will update this correction notes as soon as we find how to fix Theorem 2·2.

References

[1] J. H. PRZYTYCKI, M. V. SOKOLOV. Surgeries on periodic links and homology of periodic 3-manifolds, *Math. Proc. Cambridge Phil. Soc.(2001), Vol. 131, pp 295–307.*

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